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A BAYESIAN RELIABILITY GROWTH MODEL

by

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61

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ABSTRACT:

A model is presented for the reliability growth of a system during a test program. Parameters of the model are assumed to be random variables with appropriate prior density functions. Expressions are then derived that enable estimates (in the form of expectations) and precision statements (in the form of variances) to be made of

. projected system reliability at time τ after the start of the test program
. system reliability after the observation of failure data

Numerical examples are presented, and extension to multi-mode failures is mentioned.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. INTRODUCTION

1.1 RELIABILITY GROWTH

We are concerned with analyzing a particular model of reliability growth. The "growth" occurs in the following way: a system has some given value of a measure of reliability at the beginning of a length of time (i.e., at the start of a test period), and at the end of this period the value of this measure has changed -- hopefully, it will be improved.

This change may be caused by a number of factors. We shall be concerned, however, with only those factors that are the result of a conscious effort on the part of an interested observer (the "experimenter"). This effort is an attempt to improve or correct the system by some physical manipulation (such as component replacement or adjustment) or perhaps even by possible design change. The model considered below is similar to many discussed previously in the literature in that the corrections are attempted only after system failures have been observed.

A comparison between the model considered here (and its implications) with those contained in the literature is postponed until the final sections, where the differences in approach should become more apparent.

At this point we shall only mention the sort of information that should be, in the least, the content of any analysis of reliability growth. This content falls into two categories: inference and projection. In particular, an analysis should be able to produce statements (by necessity, probabilistic ones), on the basis of the model and the failure history to date, related to:

Inference: the present value of the reliability

Projection: the reliability at some future time, with or without continued application of the correction ("growth") process.

In order to make such statements, we shall first discuss two basic models which allow only a <u>single</u> failure mode for both discretely and continuously failing systems. This condition will be relaxed in a later section dealing with systems having many failure modes.

A final comment about the use of the word "system". As used in this paper, it shall mean simply a piece of equipment that has an assigned task to perform. If it does not perform it, it is said to have "failed". The system can be very simple, containing perhaps only one component. Or it can be extremely complicated. The only characteristic we shall use to distinguish between those degrees of complexity is the number of different (identifiable) ways it can stop functioning: i.e., the number of failure modes.

1.2 NOTATION

The following notation will be used in the description and analysis of the model discussed above:

- .Capital letters stand for events or states of nature.
- .An underlined variable, e.g., $\underline{x}$, is a random variable.
- .$f_{\underline{x}}(x)$ = p.d.f. of the r.v $\underline{x} \equiv \lim\limits_{\Delta x \to 0} \dfrac{\text{prob.}\{x \le \underline{x} \le x + \Delta x\}}{\Delta x}$
- .$\delta(x)$ = Dirac delta function* of x.

*Defined most conveniently as the limit: $\delta(x) = \lim\limits_{\epsilon \to 0} [h(x,\epsilon)]$ where

$$h(x,\epsilon) = \begin{cases} \frac{1}{\epsilon} & 0 \le x \le \epsilon \\ 0 & \text{otherwise} \end{cases}$$

$\cdot P(A|B)$ = prob. {event A given event B has occurred}.

$\cdot f_{\underline{x}}(x|A)$ = p.d.f. of $\underline{x}$ given A has occurred.

$$\equiv \lim_{\Delta x \to 0} \frac{\text{prob. } \{x \le \underline{x} \le x + dx|A\}}{\Delta x}$$

$\cdot E(\underline{x}|A) = \int x f_{\underline{x}}(x|A)dx$ = conditional expectation of $\underline{x}$ given A.

$\cdot V(\underline{x}|A) = \int [x - E(\underline{x}|A)]^2 f_{\underline{x}}(x|A)dx$ = conditional variance of $\underline{x}$ given A.

.The letter H will be used to denote the event (state of nature) "historical experience": all the prior knowledge that is available concerning the model, values of parameters of the model, etc. Probabilities and p.d.f.'s conditioned only upon H are called "a priori", or "prior".

.A vector is noted by an arrow over it, with the vector dimension being indicated in parentheses, e.g., $\vec{t}(n) = (t_1, t_2, t_3, \ldots t_n)$.

2. THE CONTINUOUS MODEL

2.1 DESCRIPTION

The system has a single failure mode, and the time between failures, $\underline{t}$, is a random variable (r.v.) with probability density function (p.d.f.)

$$f_{\underline{t}}(t) = re^{-rt} \qquad 0 \le t \le \infty .$$

The parameter r is commonly called the failure rate of the system (or, more properly, of the particular mode of failure). Since all relevant measures of reliability for an exponentially failing system can be obtained from the failure rate, it will be sufficient to concentrate upon its characteristics

only. The exponential function is not as restrictive as it may seem at first. Although it is certainly a simplistic assumption to make about complex systems, it becomes more valid as the systems become more elementary and serve to comprise the components of an even greater system. In addition, a conceptually simple (but laborious) extension of all the results of this paper is possible when it is postulated that r is in fact a function of time since last failure.

The system is, at any time, in one of two possible states (again, with respect to a single failure mode):

U = Unrepaired State

R = Repaired State

The numerical value of the failure rate r depends upon which state the system is in:

If the system is in the unrepaired state U, then $r = \lambda$;

If the system is in the repaired state R, then $r = \mu$.

The numbers λ and μ can be any non-negative values, and in fact μ is often zero. On the other hand, the value of μ might not be zero. Thus, although the system is said to be "repaired", it might still exhibit failures, albeit the failure rate when repaired might be quite low.

By virtue of a test program, the system changes states in the following restrictive way. After every failure, if the system is in U it 1) goes to R with probability a (the "repair probability"); or 2) remains in U with probability (1-a). If the system is in R, it remains in R with probability one.

Thus, there can be only one transition to state R; once the system is repaired, it remains so.

This repair attempt happens instantaneously, after which the system operates until the time of the next failure (this time being again a random variable with failure rate depending upon whether the system has been put into state R or has remained in state U).

The model may be represented by a two-state Markov process, as shown by the flow diagram of Figure 1. The times between the transitions indicated in the diagram are the times between failures and, thus, are controlled by the failure rate of whichever state the system is in:

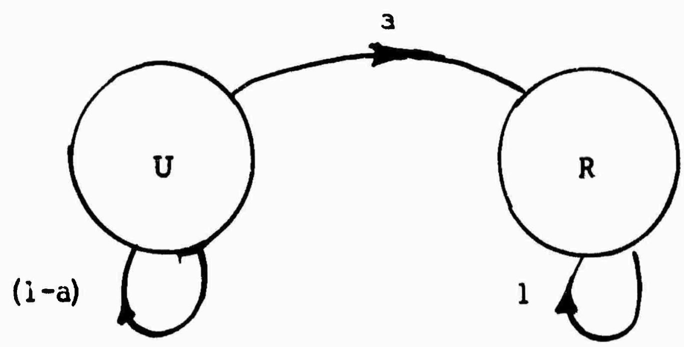

FIGURE 1

Flow diagram representation of growth model

U = Unrepaired state (failure rate = λ)

R = Repaired state (failure rate = μ)

a = repair probability

Which state the system is in, i.e., whether or not it has yet been repaired, is unknown to the observer, and he can draw conclusions as to whether or not the system is repaired only by observing the basic data: the successive failure times (or, equivalently, the times between failures).

Finally, it is possible to allow for the system to start off in a repaired state by assigning

$$p_o = \text{prob. (system is in R at the start of the test)}.$$

Except for one situation to be considered later, however, we shall always assume that $p_o = 0$.

In the above model, it is easy to see that since the system ultimately* will go to state R, if $\mu < \lambda$, the failure rate of the system will eventually decrease, and thus the reliability will grow. On the other hand, if (for some unforeseen reason) $\mu > \lambda$, it is possible to degrade the system reliability by such a test routine.

2.2 SOME BAYESIAN CONSIDERATIONS

If the numerical values of the parameters a, μ and λ, defined above, are known, then, as will be shown, it becomes a straightforward problem to make probabilistic statements about the failure rate r, at any time, on the basis of any amount of failure information. This is essentially because the value of r depends only upon the state of nature (U or R), and the transition from U to R is the extremely simple process shown in Figure 1. If the values

*As long as $a \neq 0$.

of these parameters are unknown, however, then various methods must be used in order to obtain estimates of them and then, in turn, to make statements about r. This quest is, of course, within the purview of classical statistics, and much has been written concerning the estimation of parameters of models similar to the one treated here and associated confidence intervals (see for example [1]).

The classical approach is, in essence, to 1) define some estimator (of r in this case), examine it for unbiasedness, efficiency, sufficiency, etc.; and then to 2) define an interval, the end points of which are random variables derived from the observed data, which will contain the true value of the parameter with some pre-determined probability.

The approach we choose to take is a purely inferential one. We state that before any experimentation is done the failure rates associated with states U and R are, respectively, the <u>random variables</u> $\underline{\lambda}$ and $\underline{\mu}$. (The sampling process associated with them, if one finds it necessary to imagine such, is the process of selecting a system to test from a batch of systems, the resultant picked system having associated failure rates that are thus random variables selected from the population consisting of all possible systems to be tested.)

We shall also assume that the repair probability a is known. (An obvious extension of the model results if a is also assumed to be a random variable.)

The joint probability density function of the random variables $\underline{\lambda}$ and $\underline{\mu}$, before experimentation begins, must be given, and it is assumed that this is in fact known. This (most likely subjective) prior density function is defined to be

$$f_{\underline{\lambda\mu}}(\lambda, \mu | H).$$

After some experimentation and possible correction has gone on and a series of failure times $\vec{t}(n) = (t_1, t_2, \ldots, t_n)$ has been noted, then use of the definition of conditional probability allows one to determine the "posteriori" density function.

$$f_{\underline{\lambda\mu}}(\lambda, \mu | H, \vec{t}(n)).$$

Since the failure rate of the system at any time is a function of both $\underline{\lambda}$ and $\underline{\mu}$, it is itself a random variable $\underline{r}$, with its own conditional p.d.f.

The purpose of this study is to in fact determine this density function for $\underline{r}$, both at the outset of a test period and as a function of a given set of subsequent failure times. In addition, we shall make statements concerning the density function, and its moments, for the failure rate $\underline{r}$ at any given time in the future.

2.3 KNOWN λ AND μ: RELIABILITY PROJECTION

Let us first suppose that λ and μ are deterministic and their exact numerical values are known. The failure rate $\underline{r}$ is still a random variable, however, since it depends upon whether the state of nature is U or R, and that is itself probabilistically determined. The p.d.f. for $\underline{r}$ is easily determined.

With a total test time of τ, the p.d.f. for $\underline{r}$ is $f_{\underline{r}}(r;\tau)$

$$f_{\underline{r}}(r;\tau) = \delta(r-\lambda)P(U_\tau) + \delta(r-\mu)P(R_\tau) \tag{1}$$

where

$P(U_\tau)$ = prob. {system is in U after total test time τ}

$P(R_\tau)$ = prob. {system is in R after total test time τ}

The delta function notation is used as a convenient way to write a p.d.f. for the (at this point) discrete random variable $\underline{r}$.

In what follows we assume that the system starts out in the unrepaired state P, so that $p_o = 0$. (The development can be easily extended when $p_o \neq 0$, and this will be done in a later section, where the start of the corrective testing period, $t = 0$, occurs after some previous amount of testing.)

In order to calculate $P(U_\tau) = 1 - P(R_\tau)$, we note that the event (U_τ) can be decomposed into a union of the mutually exclusive events (U_τ, F_i) where

(F_i) = event {the transition from U to R takes place on the i^{th} failure}

so that

$$(U_\tau) = \bigcup_{i=1}^{\infty}(U_\tau, F_i). \tag{2}$$

Since the F_i are mutually exclusive events, we have

$$P(U_\tau) = \sum_{i=1}^{\infty} P(U_\tau, F_i) = \sum_{i=1}^{\infty} P(U_\tau|F_i)P(F_i) \tag{3}$$

The number of the failure at which the transition from U to R takes place is geometrically distributed with parameter a, so that

$$P(F_i) = a(1-a)^{i-1} . \tag{4}$$

Furthermore, we see that

$$\begin{aligned} P(U_\tau | F_i) &= \text{prob.}\{\text{system is in U at } \tau \text{ given it goes to R at } i^{th} \text{ failure}\} \\ &= \text{prob.}\{\text{less than } i \text{ failures in time } \tau \text{ while in U}\} \\ &= \sum_{j=0}^{i-1} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} \end{aligned} \tag{5}$$

which all combine to give

$$P(U_\tau) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} a(1-a)^{i-1} \tag{6}$$

Changing the order of the summation gives

$$\begin{aligned} P(U_\tau) &= \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau} a(1-a)^{i-1} \\ &= \sum_{j=0}^{\infty} \frac{(\lambda\tau)^j}{j!} e^{-\lambda\tau}(1-a)^j = e^{-a\lambda\tau} \end{aligned} \tag{7}$$

This result can be verified by noting that the rate of transition from U to R is $a\lambda$, since

$$\begin{aligned} &\text{prob.}\{\text{transition from U to R in } \Delta\tau\} \\ &\quad = \text{prob.}\{\text{failure in } \Delta\tau | U\} \ \text{prob.}\{\text{repair}\} \\ &\quad = \lambda \Delta \tau a \end{aligned}$$

and, thus, the probability of no transition in time t is, from the Poisson

distribution, $e^{-a\lambda\tau}$. The longer derivation is useful, however, in that it indicates a technique to be used again below.

The above equations thus show that the p.d.f. of the failure rate $\underline{r}$ at time τ after start of testing is

$$f_{\underline{r}}(r;\tau) = \delta(r-\lambda)e^{-a\lambda\tau} + \delta(r-\mu)(1-e^{-a\lambda\tau}) \tag{8}$$

Note that this expression reflects a probability statement made <u>before the process starts</u>. In other words, we can interpret the quantities

$$E(\underline{r};\tau) \equiv \int_0^\infty r f_{\underline{r}}(r;\tau) = \lambda e^{-a\lambda\tau} + \mu(1-e^{-a\lambda\tau})$$

$$= \mu + (\lambda-\mu)e^{-a\lambda\tau} \tag{9}$$

and

$$V(\underline{r};\tau) \equiv \int_0^\infty [r - E(\underline{r};\tau)]^2 f_{\underline{r}}(r;\tau)$$

$$= (\lambda-\mu)^2 e^{-a\lambda\tau}(1-e^{-a\lambda\tau}) \tag{10}$$

to be the present projection of what the mean and variance of the failure rate $\underline{r}$ will be at time τ (in the future) after corrective testing.

These projections are useful in themselves as aids to reliability prediction. That is, if we know the values of the unrepaired and repaired failure rates and the value of the repair probability a, then equation (9) gives an estimate* of what the reliability will be at some time τ after testing begins, and equation (10) (actually, the square root of $V(r;\tau)$) gives an indication of the preciseness of that estimate. The behavior of these

*Optimal (i.e., cost-minimizing) for a quadratic loss function.

quantities satisfy intuition: the expectation of the failure rate starts off at λ and approaches μ. The variance starts at zero (we know $r = \lambda$ at $\tau = 0$), and returns to zero as $\tau \to \infty$ (r will certainly be equal to μ by that time, as long as $a \neq 0$), with an interesting maximum occurring at $\tau = \frac{1}{a\lambda}$.

2.4 KNOWN λ AND μ: RELIABILITY INFERENCE

All of the above analysis has been made under the consideration that the test was yet to be done. The analysis is extended now to the situation where testing has been going on for a time τ, and n failures have been observed at times $t_1, t_2, \ldots, t_n = \vec{t}(n)$, where $t_n \leq \tau < t_{n+1}$. (For ease in notation we shall now let $\vec{t} \equiv \vec{t}(n)$, with the understanding that the vector is of dimension n.)

Again, assuming still that μ and λ are deterministic and known, we would like to calculate the appropriate conditional p.d.f. for the failure rate: $f_{\underline{r}}(r|\vec{t}, \tau)$. To do so we shall need to calculate $P(R_\tau|\vec{t})$. This is shown by extending equation (1) of the preceding section,

$$f_{\underline{r}}(r|\vec{t};\tau) = \delta(r-\lambda)\,P(U_\tau|\vec{t}) + \delta(r-\mu)\,P(R_\tau|\vec{t}) \tag{11}$$

We again make use of the events F_i to write

$$\begin{aligned} P(U_\tau|\vec{t}) &= \sum_{i=1}^{\infty} P(U_\tau, F_i|\vec{t}) \\ &= \sum_{i=1}^{\infty} P(U_\tau|F_i, \vec{t})\,P(F_i|\vec{t}) \end{aligned} \tag{12}$$

But now we see that

$$P(U_\tau | F_i, \vec{t}) = \text{prob.}\ \{\text{the system is in U at } \tau \text{ given it goes to R at the } i^{th} \text{ failure, } \underline{\text{and}} \text{ failures are observed at } t_1, t_2, \ldots, t_n \ \underline{\text{and}}\ t_n \le \tau < t_{n+1}\}$$

$$= \begin{cases} 0 & \text{if } i \le n \\ 1 & \text{if } i > n \end{cases} \tag{13}$$

so that equation (12) becomes

$$P(U_\tau | \vec{t}) = \sum_{i=n+1}^{\infty} P(F_i | \vec{t}) . \tag{14}$$

Using Bayes' rule

$$P(F_i | \vec{t}) = \frac{P(\vec{t} | F_i)\, P(F_i)}{P(\vec{t})} = \frac{P(\vec{t} | F_i)\, a(1-a)^{i-1}}{P(\vec{t})} \tag{15}$$

Under the condition that $i > n$ (i.e., for all terms in the sum in equation (14)), and in fact the i^{th} failure is observed to lie between t_i and $t_i + dt_i$

$$P(\vec{t} | F_i) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2 - t_1)} \cdots \lambda e^{-\lambda(t_n - t_{n-1})} e^{-\lambda(\tau - t_n)} dt_1 dt_2 \ldots dt_n \tag{16}$$

$$= \lambda^n e^{-\lambda\tau} d\vec{t} \tag{17}$$

since the times between the first n failures, given that transition to R occurs at some failure $\underline{\text{after}}$ the n^{th}, are identically distributed exponential r.v.'s with common parameter λ. The last term in equation (17), $e^{-\lambda(\tau - t_n)}$, is due to the fact that no failures are observed in the interval (t_n, τ).

Combining this result with equations (14) and (15) yields

$$P(U_\tau | \vec{t}) = \sum_{i=n+1}^{\infty} \frac{\lambda^n e^{-\lambda\tau} a(1-a)^{i-1} d\vec{t}}{P(\vec{t})}$$

$$= \frac{\lambda^n e^{-\lambda\tau} (1-a)^n d\vec{t}}{P(\vec{t})} \tag{18}$$

We now turn our attention to calculating $P(R_\tau | \vec{t})$ in much the same fashion:

$$P(R_\tau | \vec{t}) = \sum_{i=1}^{\infty} P(R_\tau, F_i | \vec{t})$$

$$= \sum_{i=1}^{\infty} P(R_\tau | F_i, \vec{t}) \, P(F_i | \vec{t}) \tag{19}$$

Here we see that

$$P(R_\tau | F_i, \vec{t}) = \begin{cases} 1 & \text{if } i \le n \\ 0 & \text{if } i > n \end{cases} \tag{20}$$

so that

$$P(R_\tau | \vec{t}) = \sum_{i=1}^{n} P(F_i | \vec{t})$$

$$= \sum_{i=1}^{n} \frac{P(\vec{t} | F_i) \, P(F_i)}{P(\vec{t})} = \frac{\sum_{i=1}^{n} P(\vec{t} | F_i) a(1-a)^{i-1}}{P(\vec{t})} \tag{21}$$

By the same arguments that lead to equation (17) we find that, when $i \le n$

$$P(\vec{t} | F_i) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2 - t_1)} \cdots \lambda e^{-\lambda(t_i - t_{i-1})} \mu e^{-\mu(t_{i+1} - t_i)}$$

$$\cdots \mu e^{-\mu(t_n - t_{n+1})} \cdots e^{-\mu(\tau - t_n)} dt_1 dt_2 \cdots dt_n$$

$$= \lambda^i e^{-\lambda t_i} \mu^{n-i} e^{-\mu(\tau - t_i)} d\vec{t} \tag{22}$$

Using this in equation (9) gives

$$P(R_\tau | \vec{t}) = \frac{\sum_{i=1}^{n} \lambda^i e^{-\lambda t_i} \mu^{n-i} e^{-\mu(\tau - t_i)} a(1-a)^{i-1} d\vec{t}}{P(\vec{t})} \tag{23}$$

In order to evaluate $P(\vec{t})$, the common denominator in equations (18) and (23), we finally note that since (R_τ) and (U_τ) are exhaustive and mutually exclusive

$$P(R_\tau | \vec{t}) + P(U_\tau | \vec{t}) = 1$$

which, by use of equations (18) and (23) gives

$$P(U_\tau | \vec{t}) = 1 - P(R_\tau | \vec{t})$$

$$= \frac{\lambda^n e^{-\lambda\tau}(1-a)^n}{L(\vec{t}\,;\lambda,\mu)} \tag{24}$$

where the function $L(\vec{t}\,;\lambda,\mu)$ is defined to be

$$L(\vec{t}\,;\lambda,\mu) \equiv \sum_{i=1}^{n} \lambda^i e^{-\lambda t_i} \mu^{n-i} e^{-\mu(\tau - t_i)} a(1-a)^{i-1} + \lambda^n e^{-\lambda\tau}(1-a)^n$$

$$= P(\vec{t})/d\vec{t} \tag{25}$$

Combining all this with equation (11) gives, for the density function of the failure rate $\underline{r}$, having observed failures at $t_1, t_2, \ldots, t_n$ during a test period of length τ:

$$f_{\underline{r}}(r | \vec{t}; \tau) = \frac{\delta(r-\mu) \sum_{i=1}^{n} \lambda^i e^{-\lambda t_i} \mu^{n-i} e^{-\mu(\tau - t_i)} a(1-a)^{i-1} + \delta(r-\lambda)\lambda^n e^{-\lambda\tau}(1-a)^n}{L(\vec{t}\,;\lambda,\mu)} \tag{26}$$

Equations (24), (25), and (26) are the only ones necessary to make inferential statements about the reliability at time τ, given failures at times $t_1, t_2, \ldots, t_n$, and given the values of λ, μ and a.

For example, let us suppose that $\mu = 0$ (a repaired system never fails). Since

$$E(\underline{r}\,|\,\vec{t};\tau) = \int_0^\infty r\, f_{\underline{r}}(r\,|\,\vec{t};\tau)\, dr$$

we find that

$$E(\underline{r}\,|\,\vec{t};\tau) = \frac{\lambda e^{-\lambda\tau}(1-a)}{ae^{-\lambda t_n} + (1-a)e^{-\lambda\tau}} = \frac{\lambda e^{-\lambda(\tau-t_n)}}{\frac{a}{1-a} + \lambda e^{-\lambda(\tau-t_n)}} \tag{27}$$

and

$$P(U_\tau\,|\,\vec{t}) = 1 - P(R_\tau\,|\,\vec{t}) = \frac{e^{-\lambda(\tau-t_n)}}{\frac{a}{1-a} + e^{-\lambda(\tau-t_n)}} \tag{28}$$

In this case it becomes apparent that inferential statements can be made with only the information consisting of the length of time since the <u>last</u> failure $(\tau-t_n)$. This, of course, is intuitively clear, since, if $\mu = 0$, at the time of the last failure the system couldn't possibly have been repaired.

2.5 UNKNOWN λ AND μ: RELIABILITY PROJECTION

We come now to the more interesting and practical situation: that where the parameters λ and μ of the process are unknown at the start of the testing. Inferential statements about the values of these will come in

the next section. Here we will be concerned with only deriving predictive statements analagous to those implied by equations (9) and (10).

The basic technique used here is to simply consider λ and μ to be random variables $\underline{\lambda}$ and $\underline{\mu}$, with respective p.d.f.'s $f_{\underline{\lambda}}(\lambda|H)$ and $f_{\underline{\mu}}(\mu|H)$, or possibly a joint p.d.f. $f_{\underline{\lambda}\underline{\mu}}(\lambda,\mu|H)$. These *a priori* density functions are, at least at the start of experimentation, most probably subjective ones. That is, they represent all information available, at the time, relevant to the failure rates in question and expressed in terms of an appropriate density function*. If some quantitative information is available, from previous tests, etc., then of course these density functions should be conditioned not only upon the event H, but all other observed relevant data.

As a first step, we re-write equation (8) with the notation expanded to emphasize the fact that $\underline{\lambda}$ and $\underline{\mu}$ are, in that equation, deterministic and have known values λ and μ, respectively. In other words,

$$f_{\underline{r}}(r;\tau,\lambda,\mu) \equiv f_{\underline{r}}(r;\tau,\underline{\lambda}=\lambda,\underline{\mu}=\mu)$$

so that

$$f_{\underline{r}}(r;\tau,\lambda,\mu) = \delta(r-\lambda)e^{-a\lambda\tau} + \delta(r-\mu)\left(1-e^{-a\lambda\tau}\right) \tag{29}$$

We now use the well-known fact that for any probability that is itself conditioned so that it is a function of a realization of a r.v., i.e.,

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*The best techniques for producing such subjective functions are, and will probably always be, subject to a great deal of controversy. We side-step these philosophical issues here. The interested reader is referred to the copious literature on the subject, for example [7].

$P(A|\underline{x} = x)$, the unconditioned probability is simply the expectation of the conditioned one, i.e.,

$$P(A) = \int_{-\infty}^{\infty} P(A|\underline{x} = x)\, f_{\underline{x}}(x)\, dx \, * \tag{30}$$

Using this relation, we may write in place of equation (8)

$$f_{\underline{r}}(r;\tau) = \int_0^{\infty}\int_0^{\infty} f_{\underline{r}}(r;\tau,\lambda,\mu)\, f_{\underline{\lambda\mu}}(\lambda,\mu|H)\, d\lambda\, d\mu \; .$$

In all that follows we shall assume that $\underline{\lambda}$ and $\underline{\mu}$ are independent, for ease of notation, so that we may write

$$f_{\underline{\lambda\mu}}(\lambda,\mu) = f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu) \; .$$

The discussion, however, can be easily extended to the case when they are dependent variables. We shall, for convenience, also drop the conditioning event H, since all statements that can be made are all eventually conditioned upon prior experience.

Performing the indicated integration, we find

$$f_{\underline{r}}(r;\tau) = \int_0^{\infty}\int_0^{\infty} \cdot [\delta(r-\lambda)e^{-a\lambda\tau} + \delta(r-\mu)\,(1-e^{-a\lambda\tau})]\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\lambda\, d\mu$$

$$= f_{\underline{\lambda}}(r)e^{-ar\tau} + f_{\underline{\mu}}(r)\int_0^{\infty}(1-e^{-a\xi\tau})\, f_{\underline{\lambda}}(\xi)\, d\xi \tag{31}$$

from which we may derive

$$E(r;\tau) = \int_0^{\infty}\xi\, f_{\underline{\lambda}}(\xi)e^{-a\xi\tau}d\xi + E(\underline{\mu})\int_0^{\infty}(1-e^{-a\xi\tau})\, f_{\underline{\lambda}}(\xi)\, d\xi \tag{32}$$

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*For example, see Parzen [11] p. 336.

An expression for $V(r;\tau)$ may also be derived, but the specific form is complicated and does not provide any easy interpretation.

As an example of the use of equation (32), consider the case where, again, μ is known and is in fact equal zero (or, equivalently, it is a r.v. with p.d.f. $f_{\underline{\mu}}(\mu) = \delta(\mu)$). Then $E(r;\tau)$ becomes, from (32)

$$E(\underline{r};\tau) = \int_0^\infty \xi\, f_{\underline{\lambda}}(\xi) e^{-a\xi\tau} d\xi \tag{33}$$

The behavior of this expected value of failure rate at a time τ into the future (under the corrective test program) can be explored by selecting an appropriate form for the prior p.d.f. on $\underline{\lambda}$. For convenience, we select for this prior density function the conjugate form [12] gamma distribution

$$f_{\underline{\lambda}}(\lambda) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} & 0 \le \lambda \le \infty \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

which has the moments

$$E(\underline{\lambda}) = \frac{\alpha}{\beta}$$

$$V(\underline{\lambda}) = \frac{\alpha}{\beta^2}$$

This distribution thus has enough freedom for the fitting of a desired mean and variance by appropriate selection of the constants α and β.

Putting equation (34) into (32) yields

$$E(\underline{r};\tau) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta + a\tau)^{\alpha+1}} = \frac{\alpha}{\beta} \left(\frac{\beta}{\beta + a\tau}\right)^{\alpha+1}$$

$$= E(\underline{\lambda}) \left(1 + \frac{a\tau}{\beta}\right)^{-(\alpha+1)}$$

2.6 UNKNOWN λ AND μ: RELIABILITY INFERENCE

The problem of inferring the value of $\underline{r}$ after the observation of a data vector $t = t(n)$ is, of course, complicated by the fact that now $\underline{\lambda}$ and $\underline{\mu}$ are also random variables: A complete solution must also make inferential statements about the posterior distributions for these rates as well as for $\underline{r}$.

These statements, via the appropriate posterior density functions, may be easily made, however, by the judicial use of equation (30). For example, we note that equation (24) now should be written

$$P(U_\tau | \vec{t}\,;\ \underline{\lambda} = \lambda\,,\ \underline{\mu} = \mu) = \frac{\lambda^n e^{-\lambda\tau}(1-a)^n}{L(\vec{t};\ \lambda\,,\mu)} \tag{35}$$

The unconditional probability that the system is still in the unrepaired state becomes, using Bayes' Rule twice, and all limits of integration from 0 to ∞.

$$P(U_\tau | \vec{t}) = \int\int P(U_\tau | \vec{t}; \underline{\lambda} = \lambda\,, \underline{\mu} = \mu)\, f_{\underline{\lambda\mu}}(\lambda\,, \mu | \vec{t})\, d\lambda\, d\mu$$

$$= \int\int P(U_\tau | \vec{t}; \underline{\lambda} = \lambda\,, \underline{\mu} = \mu) \frac{L(\vec{t};\lambda\,,\mu)\, f_{\underline{\lambda\mu}}(\lambda\,,\mu)\, d\lambda\, d\mu}{\int\int L(\vec{t};\ \lambda\,,\mu)\, f_{\underline{\lambda\mu}}(\lambda\,,\mu)\, d\lambda\, d\mu}$$

$$= \frac{\int\int P(U_\tau\,, \vec{t} | \underline{\lambda} = \lambda\,, \underline{\mu} = \mu)\, f_{\underline{\lambda\mu}}(\lambda\,,\mu)\, d\lambda\, d\mu}{\int\int L(\vec{t};\lambda\,,\mu)\, f_{\underline{\lambda\mu}}(\lambda\mu)\, d\lambda\, d\mu}$$

$$= \frac{\int\int \lambda^n e^{-\lambda\tau}(1-a)^n f_{\underline{\lambda\mu}}(\lambda,\mu)\, d\lambda\, d\mu}{\int\int L(\vec{t};\lambda,\mu)\, f_{\underline{\lambda\mu}}(\lambda,\mu)\, d\lambda\, d\mu} \tag{36}$$

In addition, $P(R_\tau|\vec{t})$ may be obtained by noting that

$$= 1 - P(R_\tau|\vec{t}) \tag{37}$$

Similarly, it may be shown that the appropriate posterior density functions for the rates $\underline{\lambda}$ and $\underline{\mu}$ are

$$f_{\underline{\lambda}}(\lambda|\vec{t};\tau) = \frac{P(\vec{t}\,|\underline{\lambda}=\lambda)\, f_{\underline{\lambda}}(\lambda)}{\int P(\vec{t}\,|\underline{\lambda}=\lambda)\, f_{\underline{\lambda}}(\lambda)\, d\lambda}$$

$$= \frac{\int L(\vec{t};\mu,\lambda)\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\mu}{\int\int L(\vec{t};\mu,\lambda)\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\lambda\, d\mu} \tag{38}$$

and

$$f_{\underline{\mu}}(\mu|\vec{t};\tau) = \frac{\int L(\vec{t};\mu,\lambda)\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\lambda}{\int\int L(\vec{t};\mu,\lambda)\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\lambda\, d\mu} \tag{39}$$

where we have let $f_{\underline{\lambda\mu}}(\lambda,\mu) = f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)$ for ease of notation.

Finally, the same sort of manipulation leads to

$$f_{\underline{r}}(r|\vec{t};\tau) = \frac{\int \sum_{i=1}^{n} \lambda^i e^{-\lambda t_i} r^{n-1} e^{-r(\tau-t_i)} a(1-a)^{i-1} f_{\underline{\lambda}}(\lambda)\, d\lambda + r^n e^{-r\tau}(1-a)^n}{\int\int L(\vec{t};\lambda,\mu)\, f_{\underline{\lambda}}(\lambda)\, f_{\underline{\mu}}(\mu)\, d\lambda\, d\mu} \tag{40}$$

Although these equations seem formidable, they are extremely useful and valuable and provide all the information necessary for inferential statements about the system reliability, given an observed set of failure times.

In particular, knowledge of the expected values of the random variables $\underline{\lambda}$, $\underline{\mu}$ and $\underline{r}$, given $\vec{t}$, gives the experimenter good estimates of the value of

a) the failure rate before testing began: equation (38)

b) the eventual value of the failure rate after unlimited correctional testing: equation (39)

c) the present value of the failure rate: equation (40)

Additionally, the probability $P(R_T | \vec{t})$ that the system has in fact been repaired is given directly by equation (37).

As is common in all Bayesian inference schemes, the foregoing development is liable, with some justification, to the criticism that the results are dependent upon the particular prior distributions used: $f_{\underline{\lambda}}(\lambda)$ and $f_{\underline{\mu}}(\mu)$. This is indeed so, but the real concern should be with the sensitivity of the results to variations and/or extremes in the selection of prior functions. In particular, it is certainly possible to select the prior distributions with sufficiently large variances, so that the result of the analysis becomes relatively independent of the prior expectations.

On the other hand, if the failure rates in question are to any degree known in advance, it seems unreasonable not to allow the analyst to make use of his knowledge -- particularly for the making of projections.

3. THE DISCRETE MODEL

3.1 MODEL DESCRIPTION

A model similar to the one discussed above is now developed for the case where a system exhibits "discrete" failure behavior. That is, the system undergoes "trials", and at each trial the system either succeeds or fails. We assume that these trials are independent (the equivalent of the assumption of exponential behavior for the continuous model). A convenient and appropriate measure of reliability of the system at any time is simply $p = 1 - q$, where

p = probability {success on the next trial}

q = probability {failure on the next trial}

In order to model a reliability growth effect, we again consider the system to start in state U, from which it has probability a of making a transition to state R after every failure. We then define the probabilities

u = probability {system fails on a trial given in state U}

v = probability {system fails on a trial given in state R}

The analysis now proceeds exactly as in the preceding sections, and requires only some obvious notational changes (to account for the discrete character of the failure data) and additions.

Let:

$\vec{x} \equiv \{x_1, x_2, \ldots x_n\}$ = the observed data vector after n trials, where $x_i = 0$ or 1 as the i^{th} trial results in a failure or success, respectively

$$y_i = \sum_{k=1}^{i} x_i \quad (i = 1, 2, \ldots n) =$$ the cumulative number of successes up to and including the i^{th} trial

$$z_i = n - y_i =$$ the cumulative number of failures up to and including the i^{th} trial

3.2 KNOWN u AND v: RELIABILITY PROJECTION

We first consider the case where the failure probabilities u and v are deterministic and known. At the end of N trials, the system failure probability is the random variable $\underline{q}$, with p.d.f. $f_{\underline{q}}(q\,;N)$ given by

$$f_{\underline{q}}(q\,;N) = \delta(q-u)\,P(U_N) + \delta(q-v)\,P(R_N) \tag{42}$$

in direct analogy with equation (1), where

$P(U_N)$ = probability {system is in U after N trials}

$P(R_N)$ = probability {system is in R after N trials}

The value of $P(U_N)$ is readily calculated:

$$\begin{aligned} P(U_N) &= [\text{probability } \{\text{system not repaired after one trial}\}]^N \\ &= [1 - \text{probability } \{\text{system is repaired after one trial}\}]^N \\ &= [1 - a\,u]^N \end{aligned}$$

since all the N trials are in the U state, are independent, and a failure (with probability u) is necessary before a repair (probability a) is made. Equation (42) then becomes

$$f_{\underline{q}}(q\,;N) = \delta(q-u)(1-a\,u)^N + \delta(q-v)[1 - (1-a\,u)^N] \tag{43}$$

The expectation of the system failure probability at the end of N trials is $E(\underline{q}; N)$, where

$$E(\underline{q}; N) = \int_0^1 q\, f_{\underline{q}}(q; N)\, dq$$

$$= u(1-au)^N + v[1 - (1-au)^N]$$

$$= v + (u-v)(1-au)^N \tag{44}$$

3.3 KNOWN u AND v: RELIABILITY INFERENCE

In order to make inferential statements about the random variable $\underline{q}$ (and hence $\underline{p}$) given some data has been observed, we proceed again in a fashion similar to that used in the analysis of the continuous model. In particular, we may write for the conditional p.d.f. of $\underline{q}$, given the observed failure data vector $\vec{x}$:

$$f_{\underline{q}}(q|\vec{x}) = \delta(q-u)\, P(U_n|\vec{x}) + \delta(q-v)\, P(R_n|\vec{x}) \tag{45}$$

By defining the event G_i

(G_i) = event {the transition from state U to state R takes place immediately after the i^{th} failure}

we may first of all write

$$P(U_n|\vec{x}) = \sum_{i=1}^{\infty} P(U_n, G_i|\vec{x})$$

$$= \sum_{i=1}^{\infty} P(U_n|G_i, \vec{x})\, P(G_i|\vec{x}) \tag{46}$$

since

$$\bigcup_{i=1}^{\infty} (U_n, G_i | \vec{x}) = (U_n | \vec{x}) .$$

The definition of G_i allows us to write

$$P(U_n | G_i, \vec{x}) = \begin{cases} 0 & i \le z_n \\ 1 & i > z_n \end{cases}$$

since z_n is the total number of failures observed in the first n trials. Thus, if $i \le z_n$, the transition from U to R has taken place at or before the n^{th} trial, and the system cannot be in state U at the n^{th} trial.

Equation (46) can now be written

$$P(U_n | \vec{x}) = \sum_{i=z_{n+1}}^{\infty} P(G_i | \vec{x}) \tag{47}$$

and, using Bayes' Rule,

$$P(U_n | \vec{x}) = \frac{\sum_{i=z_{n+1}}^{\infty} P(\vec{x} | G_i) P(G_i)}{P(\vec{x})}$$

The value of $P(G_i)$ is determined from the underlying geometric process with parameter a, so that

$$P(U_n | \vec{x}) = \frac{\sum_{i=z_{n+1}}^{\infty} P(\vec{x} | G_i) a(1-a)^{i-1}}{P(\vec{x})} \tag{48}$$

We now note that when the transition from U to R takes place at some trial after the n^{th} [i.e., for all terms in the summation in equation (48)], we may write

$$P(\vec{x}\,|G_i) = u^{1-x_1}(1-u)^{x_1} u^{1-x_2}(1-u)^{x_2} \ldots u^{1-x_n}(1-u)^{x_n}$$

$$= u^{z_n}(1-u)^{y_n}$$

since all n trials take place while the system is in the U state. Combining this result with equation (48) gives

$$P(U_n|\vec{x}) = \frac{\sum_{i=z_{n+1}}^{\infty} u^{z_n}(1-u)^{y_n} a(1-a)^{i-1}}{P(\vec{x})}$$

$$= \frac{u^{z_n}(1-u)^{y_n}(1-a)^{z_n}}{P(\vec{x})} \tag{49}$$

The calculation of $P(R_n|\vec{x})$ is also accomplished by use of the exhaustive and exclusive character of the event (G_i) $i = 1, 2, \ldots \infty$.

$$P(R_n|\vec{x}) = \sum_{i=1}^{\infty} P(R_n, G_i|\vec{x})$$

$$= \sum_{i=1}^{\infty} P(R_n|G_i, \vec{x})\, P(G_i|\vec{x}) \tag{50}$$

The value of $P(R_n|G_i, \vec{x})$ is determined by the same arguments that led to equation (47):

$$P(R_n|G_i, \vec{x}) = \begin{cases} 1 & i \le z_n \\ 0 & i > z_n \end{cases} \tag{51}$$

so that equation (50) becomes

$$P(R_n | \vec{x}) = \sum_{i=1}^{z_n} P(G_i | \vec{x})$$

and, using Bayes' Rule and $P(G_i) = a(1-a)^{i-1}$,

$$P(R_n | \vec{x}) = \frac{\sum_{i=1}^{z_n} P(\vec{x} | G_i)\, a(1-a)^{i-1}}{P(\vec{x})} \qquad (52)$$

where the summation is defined to be zero when $z_n = 0$.

Finally, we note that when $i \leq z_n$

$$P(\vec{x} | G_i) = \left[u^{1-x_1}(1-u)^{x_1} u^{1-x_2}(1-u)^{x_2} \ldots u^{1-x_i}(1-u)^{x_i}\right] \times \left[v^{1-x_{i+1}}(1-v)^{x_{i+1}} \ldots v^{1-x_n}(1-v)^{x_n}\right]$$

$$= u^{i-y_i}(1-u)^{y_i} v^{n-i-y_n+y_i}(1-v)^{y_n-y_i}$$

$$= u^{z_i}(1-u)^{y_i} v^{z_n-z_i}(1-v)^{y_n-y_i} \qquad (53)$$

so that

$$P(R_n | \vec{x}) = \frac{\sum_{i=1}^{z_n} u^{z_i}(1-u)^{y_i} v^{z_n-z_i}(1-v)^{y_n-y_i} a(1-a)^{i-1}}{P(\vec{x})} \qquad (54)$$

Complete inferential statements about the failure probability q, given the observed data $\vec{x}$, may now be readily made using the posterior p.d.f. $f_{\underline{q}}(q | \vec{x})$. This has been obtained, essentially, since we now need to

simply substitute the expressions for $P(U_n|\vec{x})$ and $P(R_n|\vec{x})$ (from equations (49) and (54), respectively) into equation (45). Note that the common term of $P(\vec{x})$ in the denominators of equations (49) and (54) can be evaluated by means of

$$P(U_N|\vec{x}) + P(R_n|\vec{x}) = 1$$

3.4 UNKNOWN u AND v: RELIABILITY PROJECTION

When the failure probabilities u and v are unknown, we proceed as in section 2.5 by treating these parameters as random variables $\underline{u}$ and $\underline{v}$, with joint p.d.f. $f_{\underline{uv}}(u,v) = f_{\underline{uv}}(u,v|H)$. Again, we shall (for ease in development) assume that $\underline{u}$ and $\underline{v}$ are independent, so that

$$f_{\underline{uv}}(u,v) = f_{\underline{u}}(u)\, f_{\underline{v}}(v)$$

Use of the technique illustrated by equation (30) gives the following results. (Intermediate steps have been left out. The development parallels that of section 2.5)

$$f_{\underline{q}}(q;N) = \int_0^1\int_0^1 \{\delta(q-u)(1-au)^N + \delta(q-v)[1-(1-au)^N]\}\, f_{\underline{uv}}(u,v)\,du\,dv$$

$$= (1-aq)^N f_{\underline{u}}(q) + f_{\underline{v}}(q)\int_0^1 [1-(1-a\xi)^N]\, f_{\underline{u}}(\xi)\,d\xi \tag{55}$$

The projected expectation of the failure probability at the end of N trials is

$$E(q;N) = \int_0^1 q\, f_{\underline{q}}(q;N)\,dq$$

$$= \int_0^1 \xi\, f_{\underline{u}}(\xi)(1-a\xi)^N d\xi + E(\underline{v})\int_0^1 [1-(1-a\xi)]^N f_{\underline{v}}(\xi)\,d\xi \tag{56}$$

3.5 UNKNOWN u AND v: RELIABILITY INFERENCE

When a data vector $\vec{x}$ has been observed, and $\underline{u}$ and $\underline{v}$ are random variables with prior p.d.f. $f_{\underline{uv}}(u,v)$, conditional density functions on $\underline{u}$, $\underline{v}$ and $\underline{q}$ can be derived in a manner parallel to that used for the continuous case in section 2.6.

To keep the expressions concise, we define the following terms:

$$P(U_N, \vec{x}; u) = u^{z_n}(1-u)^{y_n}(1-a)^{z_n} \tag{57}$$

$$P(R_n, \vec{x}; u, v) = \sum_{i=1}^{z_n} u^{z_i}(1-u)^{y_i} v^{z_n - z_i}(1-v)^{y_n - y_i} a(1-a)^{i-1} \tag{58}$$

$$P(\vec{x}; u, v) = P(U_n, \vec{x}; u) + P(R_n, \vec{x}; u, v) \tag{59}$$

$$P(\vec{x}) = \int_0^1 \int_0^1 P(\vec{x}; u, v)\, f_{\underline{uv}}(u,v)\, du\, dv \tag{60}$$

The posterior density functions of interest then become (after intermediate steps similar to those in section 2.6)

$$f_{\underline{u}}(u|\vec{x}) = \frac{\int_0^1 P(\vec{x}; u, v)\, f_{\underline{uv}}(u,v)\, dv}{P(\vec{x})} \tag{61}$$

$$f_{\underline{v}}(v|\vec{x}) = \frac{\int_0^1 P(\vec{x}; u, v)\, f_{\underline{uv}}(u,v)\, du}{P(\vec{x})} \tag{62}$$

$$f_{\underline{q}}(q|\vec{x}) = \frac{\int_0^1 P(R_n, \vec{x}; u, q)\, f_{\underline{u}}(u)\, du + P(U_n; \vec{x}, q)}{P(\vec{x})} \tag{63}$$

and the posterior probability that the system has been repaired is

$$P(R_n \mid \vec{x}) = \frac{\int_0^1 \int_0^1 P(R_n, \vec{x}; u, v)\, f_{\underline{uv}}(u,v)\, du\, dv}{P(\vec{x})} \tag{64}$$

4. NUMERICAL EXAMPLES

4.1 CONTINUOUS MODEL

A numerical example is now presented to illustrate the use of the results of the previous sections.

The first task is the assignment of appropriate prior probability density functions for the failure rates $\underline{\lambda}$ (before repair) and $\underline{\mu}$ (after repair). In order to facilitate calculations it is convenient to assume that these random variables are independent and have prior density functions of the Gamma family, so that

$$f_{\underline{\lambda}}(\lambda) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \lambda^{\alpha_1 - 1} e^{-\beta_1 \lambda} \tag{65}$$

$$f_{\underline{\mu}}(\mu) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \mu^{\alpha_2 - 1} e^{-\beta_2 \mu} \tag{66}$$

Furthermore, we suppose that estimates are available for the moments of $\underline{u}$ and $\underline{v}$. A particular set of such estimates is

$$\begin{array}{ll} E(\underline{\lambda}) = 1 & E(\underline{\mu}) = .5 \\ \sigma(\underline{\lambda}) = 1 & \sigma(\underline{\mu}) = .5 \end{array} \tag{67}$$

where $E(\underline{\lambda}) = \int_0^1 \lambda \, f_{\underline{\lambda}}(\lambda) \, d\lambda$ = expected value of $\underline{\lambda}$

$$V(\lambda) = \sigma^2(\underline{\lambda}) = \int_0^1 [\lambda - E(\underline{\lambda})]^2 f_{\underline{\lambda}}(\lambda) \, d\lambda = \text{variance of } \underline{\lambda}$$

This set of estimates, in conjunction with equations (65) and (66) give

$\alpha_1 = 1$ $\qquad\qquad$ $\alpha_2 = 1$

$\beta_1 = 1$ $\qquad\qquad$ $\beta_2 = 2$

The repair probability is assumed known and to have value $a = .25$.

These figures are selected not with a physical example in mind, but with the intention of displaying the underlying features of the model. Thus we at this point have assumed the following.

. At the start of testing, the system has a constant failure rate λ that is unknown, but is estimated to be about 1 (per unit time). The precision of this estimate is indicated by a standard deviation of 1 (per unit time).

. After every failure an attempt at repair is made. This attempt has probability $a = .25$ of succeeding, i.e., putting the system in the "repaired" state.

. When the system has been repaired, the failure rate decreases to a constant value μ which is unknown, but which (from experience or judicial guessing) can be estimated to be .5 (per unit time) with a standard deviation also of .5 (per unit time).

We now proceed to make statements about: the failure rate after some length of future test time (projection); updated estimates of λ and μ on

the basis of failure data gathered during the experiment (inference); the system failure rate r after observation of failure data.

<u>Projection:</u>

Using the values given above, the p.d.f. for the failure rate $\underline{r}$ at some time τ after the start of the growth program is, from equation (31)

$$f_{\underline{r}}(r;\tau) = e^{-r(1+.25\tau)} + \frac{.5\tau e^{-2r}}{1+.25\tau} \tag{68}$$

and so the expected value of the failure rate after time τ is, from (32)

$$E(\underline{r}) = \left(\frac{1}{1+.25\tau}\right)^2 + \frac{.5\tau}{(1+.25\tau)} \tag{69}$$

From this expression we see that the expected failure rate will drop halfway between its unrepaired and repaired values after a length of approximately $\tau \approx 12$ units.

<u>Inference:</u>

In order to make inferential statements about $\underline{\lambda}$, $\underline{\mu}$ and $\underline{r}$, a data vector is needed.

Suppose that failures are observed, after the start of testing, at times 1, 2, 3, 4, 6.2, 8.2, 10.2, so that n = number of failures = 7 and

$\vec{t} = (1,2,3,4,6.2,8.2,10.2)$

[This data vector was chosen to intentionally -- and crudely -- simulate a "repair" at t = 4 and a decrease in failure rate from 1 to .5]

For any time τ, equations (38), (39) and (40) give the p.d.f. for $\underline{\lambda}$, $\underline{\mu}$ and $\underline{r}$, respectively; equation (36) gives the probability that the system

has been repaired at or before that time. In our numerical example, we can examine these posterior density functions by finding their means and standard deviations. For the prior parameters and data vector given above, these have been calculated and are shown in Table 1 for values of τ from 0 to 10.2 by increments of $\Delta\tau = .2$ time units.

Projection after Inference:

At this point it is possible to extend the development to describe the following situation.

Suppose that prior parameters have been selected, as above, and the inferential calculations carried out. At time $\tau = 10.2$, after having seen the 7 failures described by $\vec{t}$, what can we say about the expectation of the failure rate at some time τ' after time $\tau = 10.2$?

In order to answer this question we note that at time $\tau = 10.2$ we have (see Table 1)

$$E(\underline{\lambda}) = .917 \qquad E(\underline{\mu}) = .543$$

$$\sigma(\underline{\lambda}) = .522 \qquad \sigma(\underline{\mu}) = .322 \tag{70}$$

$$P(R_{12} | \vec{t}) = .846$$

We are now faced with the situation described in the discussion following equation (1). For we may consider the situation to be such that the values of equation (70) describe our total knowledge about $\underline{\lambda}$ and $\underline{\mu}$ up to that point; i.e., they can serve to define a new "prior" density function, with parameters α_1', β_1', α_2' and β_2'.

FAILURES	TIME	$E(\lambda)$	$\sigma(\lambda)$	$E(\mu)$	$\sigma(\mu)$	$P(R_T)$	$E(r)$	$\sigma(r)$
0	.20	.833	.833	.500	.500			
0	.40	.714	.714	.500	.500			
0	.60	.625	.625	.500	.500			
0	.80	.556	.556	.500	.500			
1	1.00	1.000	.707	.500	.500	.250	.875	.696
1	1.20	.933	.662	.488	.489	.268	.814	.651
1	1.40	.881	.630	.476	.479	.286	.765	.618
1	1.60	.839	.607	.465	.471	.302	.726	.595
1	1.80	.805	.592	.455	.464	.318	.694	.578
2	2.00	1.000	.606	.533	.499	.400	.813	.610
2	2.20	.964	.588	.516	.483	.420	.776	.589
2	2.40	.934	.575	.500	.469	.439	.743	.573
2	2.60	.910	.567	.484	.456	.457	.715	.561
2	2.80	.890	.562	.470	.446	.475	.690	.551
3	3.00	1.000	.558	.574	.495	.510	.783	.568
3	3.20	.978	.549	.554	.477	.528	.754	.554
3	3.40	.959	.543	.535	.461	.545	.728	.543
3	3.60	.943	.540	.517	.447	.562	.704	.533
3	3.80	.930	.537	.500	.435	.578	.681	.526
4	4.00	1.000	.534	.616	.486	.598	.770	.540
4	4.20	.985	.529	.594	.469	.612	.746	.528
4	4.40	.972	.526	.573	.453	.627	.722	.519
4	4.60	.962	.523	.554	.439	.641	.700	.510
4	4.80	.952	.522	.535	.426	.655	.679	.502
4	5.00	.945	.521	.517	.415	.669	.659	.496
4	5.20	.939	.521	.500	.404	.683	.639	.489
4	5.40	.933	.521	.484	.394	.696	.620	.483
4	5.60	.929	.522	.468	.385	.708	.602	.478
4	5.80	.926	.522	.452	.377	.721	.585	.473
4	6.00	.924	.523	.438	.368	.733	.568	.468
5	6.20	.939	.515	.566	.404	.720	.670	.469
5	6.40	.934	.515	.549	.393	.730	.653	.462
5	6.60	.930	.515	.533	.383	.740	.636	.456
5	6.80	.926	.515	.517	.374	.749	.620	.450
5	7.00	.922	.515	.502	.366	.758	.604	.445
5	7.20	.920	.515	.488	.358	.767	.588	.440
5	7.40	.917	.515	.474	.351	.776	.573	.435
5	7.60	.915	.515	.460	.344	.784	.558	.430
5	7.80	.914	.515	.447	.337	.793	.544	.425
5	8.00	.912	.515	.435	.331	.801	.530	.420
6	8.20	.923	.517	.552	.359	.794	.628	.424
6	8.40	.920	.517	.537	.350	.800	.614	.418
6	8.60	.918	.517	.523	.342	.806	.600	.413
6	8.80	.916	.517	.510	.335	.812	.586	.408
6	9.00	.914	.516	.497	.329	.819	.573	.403
6	9.20	.912	.516	.485	.322	.825	.560	.399
6	9.40	.910	.516	.473	.317	.830	.547	.394
6	9.60	.909	.515	.462	.311	.836	.535	.390
6	9.80	.908	.514	.450	.306	.842	.523	.386
6	10.00	.906	.514	.439	.301	.847	.511	.381
7	10.20	.917	.522	.543	.322	.846	.601	.385

TABLE 1

Doing so, we find that

$$\alpha_1' = 1.75 \qquad\qquad \alpha_2' = 1.68$$

$$\beta_1' = 1.92 \qquad\qquad \beta_2' = 3.10$$

In addition, we now have the situation where the value of

$$p_o = \text{prob }\{\text{system is in R at time } 0\}$$

$$= P\{R_{12} \mid \vec{t}\} = .846$$

A simple argument leads to the modification of equation (8) for the case when $p_o \neq 0$:

$$f_{\underline{r}}(r;\tau) = \delta(r-\lambda)(1-p_o)e^{-a\lambda\tau} + \delta(r-\mu)[1-(1-p_o)e^{-a\lambda\tau}] \tag{71}$$

and, consequently, equation (31) becomes

$$f_{\underline{r}}(r;\tau) = (1-p_o)f_{\underline{\lambda}}(r)e^{-ar\tau} + f_{\underline{\mu}}(r)\int_0^\infty [1-(1-p_o)e^{-a\xi\tau}]f_{\underline{\lambda}}(\xi)d\xi \tag{72}$$

Taking the expectation of equation (72), using the primed prior parameters, we get

$$E(\underline{r} \mid \vec{t};\tau') = \text{expected value of failure rate time } \tau' \text{ after } \tau = 12, \text{ given } \vec{t}$$

$$= (1-p_o)\frac{\alpha_1'}{\beta_1'}\left(\frac{\beta_1'}{\beta_1'+a\tau'}\right)^{\alpha_1'+1} + \frac{\alpha_2}{\beta_2}\left[1-(1-p_o)\left(\frac{\beta_1'}{\beta_1'+a\tau'}\right)^{\alpha_1'}\right]$$

$$= .543 + \frac{.485(.72 - .136\tau')}{(1.92 + .25\tau')^{2.75}}$$

Sensitivity:

The model has not been fully evaluated with regard to the sensitivity of results to values of the prior parameters, errors in estimation of a, etc. However, examples for various cases have been calculated.

Tables 3 through 6 show $E(\lambda)$, $\sigma(\lambda)$, $E(\mu)$, $\sigma(\mu)$, $P(R_\tau)$, $E(r)$ and $\sigma(r)$ all conditioned upon the data vector $\vec{t} = (1,2,3,4,6.2,8.2,10.2)$ and evaluated at $\tau = 0$ to 10.2 by increments of $\Delta\tau = .2$ time units. These calculations contain the prior parameters as shown in Table 2.

Table	α_1	β_1	α_2	β_2	$E(\lambda)$	$\sigma(\lambda)$	$E(\mu)$	$\sigma(\mu)$	a
1	1	1	1	2	1	1	.5	.5	.25
3	4	4	4	8	1	.5	.5	.25	.25
4	1	2	1	4	.5	.5	.25	.25	.25
5	4	4	4	8	1	.5	.5	.25	.12
6	4	4	4	8	1	.5	.5	.25	.50

TABLE 2

Prior Parameters Used in Calculations of Tables 3-6

4.2 DISCRETE MODEL

For the discrete model, numerical calculations become simplified when the prior probability density functions for the failure probabilities $\underline{\mu}$ and $\underline{v}$ are of the Beta family of p.d.f.'s, where

$$B(x;\alpha,\beta) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} x^{\alpha-1}(1-x)^{\beta-\alpha-1} \tag{73}$$

FAILURES	TIME	E(λ)	σ(λ)	E(μ)	σ(μ)	$P(R_T)$	E(r)	σ(r)
0	.20	.952	.476	.500	.250			
0	.40	.909	.455	.500	.250			
0	.60	.870	.435	.500	.250			
0	.80	.833	.417	.500	.250			
1	1.00	1.000	.447	.500	.250	.250	.875	.461
1	1.20	.972	.435	.497	.248	.269	.844	.446
1	1.40	.947	.425	.493	.247	.287	.817	.434
1	1.60	.926	.417	.489	.245	.305	.792	.424
1	1.80	.907	.412	.485	.244	.323	.770	.418
2	2.00	1.000	.417	.512	.251	.412	.799	.431
2	2.20	.982	.410	.507	.248	.435	.775	.421
2	2.40	.966	.406	.502	.245	.457	.754	.413
2	2.60	.953	.402	.497	.243	.478	.735	.405
2	2.80	.941	.400	.492	.240	.498	.717	.399
3	3.00	1.000	.398	.520	.252	.529	.751	.404
3	3.20	.988	.394	.524	.249	.551	.732	.396
3	3.40	.977	.392	.518	.245	.571	.715	.390
3	3.60	.968	.391	.512	.242	.591	.698	.384
3	3.80	.960	.390	.506	.239	.610	.683	.378
4	4.00	1.000	.386	.550	.253	.618	.722	.380
4	4.20	.991	.384	.544	.249	.637	.706	.373
4	4.40	.984	.382	.537	.246	.656	.691	.367
4	4.60	.977	.382	.531	.242	.673	.677	.362
4	4.80	.972	.382	.524	.238	.689	.663	.357
4	5.00	.967	.382	.517	.235	.705	.650	.352
4	5.20	.963	.382	.510	.232	.720	.637	.348
4	5.40	.960	.383	.503	.229	.734	.625	.344
4	5.60	.958	.384	.496	.225	.747	.613	.340
4	5.80	.956	.385	.489	.222	.759	.601	.336
4	6.00	.954	.386	.482	.220	.771	.590	.333
5	6.20	.964	.376	.538	.232	.763	.639	.328
5	6.40	.961	.377	.531	.229	.775	.628	.324
5	6.60	.959	.378	.524	.225	.786	.617	.320
5	6.80	.957	.378	.517	.222	.796	.607	.316
5	7.00	.955	.379	.510	.219	.806	.596	.312
5	7.20	.954	.380	.503	.216	.815	.586	.309
5	7.40	.953	.381	.496	.213	.824	.576	.305
5	7.60	.952	.381	.489	.211	.832	.567	.302
5	7.80	.952	.382	.482	.208	.840	.557	.299
5	8.00	.951	.383	.475	.205	.847	.548	.296
6	8.20	.953	.377	.534	.216	.841	.601	.292
6	8.40	.952	.377	.527	.213	.848	.591	.289
6	8.60	.951	.378	.520	.210	.855	.582	.286
6	8.80	.951	.378	.513	.208	.861	.574	.283
6	9.00	.950	.379	.506	.205	.868	.565	.280
6	9.20	.950	.380	.500	.202	.873	.557	.276
6	9.40	.949	.380	.493	.200	.879	.548	.273
6	9.60	.949	.380	.487	.198	.884	.540	.270
6	9.80	.949	.381	.480	.195	.889	.532	.268
6	10.00	.948	.381	.474	.193	.894	.524	.265
7	10.20	.949	.378	.530	.202	.890	.576	.263

3

TABLE 3

38

FAILURES	TIME	E(λ)	σ(λ)	E(μ)	σ(μ)	$P(R_T)$	E(r)	σ(r)
0	.20	.455	.455	.250	.250			
0	.40	.417	.417	.250	.250			
0	.60	.385	.385	.250	.250			
0	.80	.357	.357	.250	.250			
1	1.00	.667	.471	.250	.250	.250	.563	.464
1	1.20	.636	.450	.247	.247	.265	.533	.441
1	1.40	.610	.434	.244	.244	.280	.508	.423
1	1.60	.588	.421	.240	.241	.294	.486	.409
1	1.80	.570	.411	.237	.239	.308	.467	.397
2	2.00	.737	.440	.274	.261	.370	.566	.444
2	2.20	.714	.428	.270	.256	.390	.541	.430
2	2.40	.695	.419	.265	.252	.409	.519	.417
2	2.60	.679	.411	.261	.247	.428	.500	.407
2	2.80	.666	.406	.256	.243	.446	.483	.399
3	3.00	.775	.418	.304	.276	.450	.563	.430
3	3.20	.758	.410	.299	.270	.471	.542	.419
3	3.40	.744	.403	.293	.264	.492	.522	.409
3	3.60	.731	.399	.288	.258	.511	.505	.401
3	3.80	.721	.396	.282	.253	.531	.488	.394
4	4.00	.790	.402	.337	.292	.515	.561	.418
4	4.20	.785	.396	.331	.285	.535	.542	.409
4	4.40	.774	.392	.325	.278	.555	.525	.401
4	4.60	.764	.388	.319	.271	.574	.508	.394
4	4.80	.756	.386	.312	.265	.593	.493	.387
4	5.00	.750	.385	.306	.259	.611	.478	.381
4	5.20	.744	.384	.299	.254	.628	.464	.376
4	5.40	.740	.384	.292	.248	.645	.451	.371
4	5.60	.736	.384	.286	.243	.661	.438	.367
4	5.80	.734	.384	.279	.238	.677	.426	.362
4	6.00	.732	.385	.273	.233	.692	.414	.358
5	6.20	.747	.372	.350	.270	.643	.491	.364
5	6.40	.742	.371	.343	.264	.658	.479	.359
5	6.60	.737	.371	.336	.258	.672	.468	.354
5	6.80	.734	.370	.329	.253	.686	.456	.349
5	7.00	.731	.370	.323	.248	[illegible]	.445	.345
5	7.20	.728	.370	.316	.243	.712	.435	.341
5	7.40	.726	.370	.309	.238	.725	.424	.337
5	7.60	.725	.371	.303	.233	.737	.414	.333
5	7.80	.724	.371	.296	.229	.748	.404	.329
5	8.00	.723	.371	.290	.225	.759	.394	.325
6	8.20	.729	.365	.369	.256	.722	.469	.333
6	8.40	.726	.365	.362	.251	.732	.459	.328
6	8.60	.724	.365	.355	.246	.743	.450	.324
6	8.80	.722	.365	.349	.242	.753	.441	.321
6	9.00	.720	.365	.342	.237	.763	.432	.317
6	9.20	.719	.365	.335	.233	.772	.423	.313
6	9.40	.718	.365	.329	.229	.781	.414	.310
6	9.60	.717	.365	.322	.225	.790	.405	.306
6	9.80	.716	.365	.316	.221	.799	.396	.303
6	10.00	.715	.365	.310	.218	.807	.388	.299
7	10.20	.720	.365	.387	.243	.732	.459	.307

TABLE 4

FAILURES	TIME	$E(\lambda)$	$\sigma(\lambda)$	$E(\mu)$	$\sigma(\mu)$	$P(R_T)$	$E(r)$	$\sigma(r)$
0	.20	.952	.476	.500	.250			
0	.40	.909	.455	.500	.250			
0	.60	.870	.435	.500	.250			
0	.80	.833	.417	.500	.250			
1	1.00	1.000	.447	.500	.250	.120	.940	.458
1	1.20	.967	.432	.498	.249	.131	.905	.442
1	1.40	.936	.420	.497	.248	.142	.874	.428
1	1.60	.909	.409	.495	.248	.153	.846	.416
1	1.80	.885	.400	.493	.247	.164	.820	.406
2	2.00	1.000	.412	.506	.250	.209	.897	.434
2	2.20	.975	.403	.503	.249	.225	.869	.422
2	2.40	.953	.395	.501	.248	.241	.844	.413
2	2.60	.932	.388	.498	.246	.257	.821	.405
2	2.80	.914	.383	.495	.245	.273	.800	.398
3	3.00	1.000	.388	.514	.251	.280	.864	.416
3	3.20	.981	.381	.511	.249	.299	.840	.408
3	3.40	.963	.376	.508	.247	.317	.819	.401
3	3.60	.947	.372	.505	.245	.335	.799	.395
3	3.80	.934	.369	.501	.244	.353	.781	.390
4	4.00	1.000	.369	.524	.253	.340	.839	.403
4	4.20	.984	.365	.521	.250	.359	.818	.396
4	4.40	.970	.361	.518	.248	.378	.799	.391
4	4.60	.958	.359	.514	.246	.397	.782	.386
4	4.80	.947	.357	.510	.243	.416	.765	.381
4	5.00	.937	.356	.506	.241	.434	.750	.378
4	5.20	.929	.356	.502	.239	.452	.736	.375
4	5.40	.921	.356	.497	.237	.470	.722	.372
4	5.60	.915	.357	.493	.234	.488	.709	.370
4	5.80	.909	.358	.488	.232	.505	.696	.368
4	6.00	.904	.360	.483	.230	.522	.684	.366
5	6.20	.931	.345	.519	.240	.480	.734	.363
5	6.40	.924	.345	.515	.237	.496	.721	.360
5	6.60	.918	.346	.511	.235	.513	.709	.358
5	6.80	.913	.347	.506	.232	.529	.697	.355
5	7.00	.908	.348	.501	.230	.545	.686	.353
5	7.20	.904	.349	.496	.228	.561	.675	.352
5	7.40	.901	.351	.491	.226	.576	.665	.350
5	7.60	.898	.352	.486	.223	.590	.655	.348
5	7.80	.896	.354	.481	.221	.605	.645	.347
5	8.00	.894	.356	.475	.219	.619	.635	.345
6	8.20	.905	.340	.517	.229	.573	.683	.341
6	8.40	.901	.342	.513	.227	.587	.673	.339
6	8.60	.898	.343	.508	.224	.601	.664	.337
6	8.80	.895	.345	.503	.222	.614	.654	.336
6	9.00	.893	.346	.498	.220	.627	.645	.334
6	9.20	.891	.348	.493	.217	.640	.636	.332
6	9.40	.890	.349	.488	.215	.652	.628	.331
6	9.60	.888	.351	.483	.213	.664	.619	.329
6	9.80	.887	.353	.477	.211	.675	.610	.328
6	10.00	.886	.354	.472	.209	.686	.602	.326
7	10.20	.891	.341	.516	.219	.646	.648	.323

5

TABLE 5

6

FAILURES	TIME	$E(\lambda)$	$\sigma(\lambda)$	$E(\mu)$	$\sigma(\mu)$	$P(R_\tau)$	$E(r)$	$\sigma(r)$
0	.20	.952	.476	.500	.250			
0	.40	.909	.455	.500	.250			
0	.60	.870	.435	.500	.250			
0	.80	.833	.417	.500	.250			
1	1.00	1.000	.447	.500	.250	.500	.750	.440
1	1.20	.982	.440	.494	.247	.524	.726	.428
1	1.40	.966	.434	.487	.244	.547	.704	.418
1	1.60	.954	.430	.480	.241	.569	.684	.410
1	1.80	.943	.428	.473	.238	.589	.666	.403
2	2.00	1.000	.426	.525	.251	.727	.655	.374
2	2.20	.992	.424	.517	.246	.744	.638	.366
2	2.40	.985	.422	.508	.242	.761	.622	.359
2	2.60	.980	.421	.500	.238	.775	.607	.352
2	2.80	.975	.421	.491	.234	.789	.593	.346
3	3.00	1.000	.417	.558	.251	.846	.626	.325
3	3.20	.996	.416	.548	.246	.857	.612	.318
3	3.40	.993	.416	.539	.242	.867	.599	.312
3	3.60	.990	.416	.529	.237	.876	.587	.306
3	3.80	.988	.416	.520	.233	.884	.574	.301
4	4.00	1.000	.414	.593	.250	.914	.628	.291
4	4.20	.998	.413	.582	.245	.920	.616	.286
4	4.40	.996	.413	.572	.240	.925	.604	.281
4	4.60	.995	.413	.562	.236	.930	.592	.276
4	4.80	.994	.413	.552	.232	.935	.581	.271
4	5.00	.993	.413	.543	.228	.939	.570	.266
4	5.20	.992	.413	.533	.224	.942	.560	.262
4	5.40	.992	.413	.524	.221	.946	.550	.258
4	5.60	.992	.413	.515	.217	.949	.540	.254
4	5.80	.991	.413	.507	.214	.952	.530	.250
4	6.00	.991	.413	.498	.211	.955	.521	.246
5	6.20	.992	.412	.570	.222	.967	.584	.243
5	6.40	.992	.412	.561	.219	.969	.574	.240
5	6.60	.992	.412	.552	.216	.970	.565	.236
5	6.80	.992	.412	.543	.212	.972	.556	.232
5	7.00	.991	.412	.535	.209	.974	.547	.229
5	7.20	.991	.412	.526	.206	.975	.538	.226
5	7.40	.991	.412	.518	.203	.976	.530	.223
5	7.60	.991	.412	.511	.201	.977	.521	.220
5	7.80	.991	.411	.503	.198	.979	.513	.217
5	8.00	.991	.411	.495	.195	.980	.506	.214
6	8.20	.991	.413	.561	.204	.986	.567	.215
6	8.40	.991	.413	.553	.201	.986	.559	.212
6	8.60	.991	.412	.545	.199	.987	.551	.209
6	8.80	.991	.412	.537	.196	.988	.543	.207
6	9.00	.991	.412	.530	.194	.988	.535	.204
6	9.20	.991	.412	.523	.191	.989	.528	.201
6	9.40	.991	.412	.516	.189	.989	.521	.199
6	9.60	.990	.411	.509	.187	.990	.514	.196
6	9.80	.990	.411	.502	.184	.990	.507	.194
6	10.00	.990	.411	.495	.182	.991	[illegible]	.192
7	10.20	.991	.413	.554	.190	.992	.557	.195

TABLE 6

41

The moments of this function are

$$E(\underline{x}) = \alpha/\beta$$

$$V(\underline{x}) = \sigma^2(\underline{x}) = \frac{\alpha}{\beta}(1 - \frac{\alpha}{\beta}) \frac{1}{\beta+1} \tag{74}$$

Unfortunately, even this usually "conjugate prior" form does not allow a closed form solution of the projection problem, as exemplified in equations (55) and (56). This is <u>not</u> to say that specific projections cannot be made -- the associated numerical integrations are straightforward, but have not been attempted here.

The more interesting inferential problem may be easily evaluated, however, and is illustrated in Tables 8 through 12.

The data vector is assumed to be

$$\vec{x} = (0,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,1,1,0,1,1,1,0)$$

where a "0" represents a failure, a "1" represents a success. Again, this "observed" data vector has been pre-selected to simulate an overly typical result that might appear if $u = .5$ $v = .25$ and repair took place on the 7th trial (the 4th failure). Numerical results now simply require a set of prior parameters and the determination of the first and second moments of equations (61), (62) and (63).

In the calculation of a number of cases for various values of prior parameters, it becomes convenient to work with the success probabilities $1-u$ and $1-v$, rather than u and v directly. Table 7 shows the selection of values of the prior parameters for $1-u$ and $1-v$, and for the repair probability a.

Table	E(1-u)	σ(1-u)	E(1-v)	σ(1-v)	a
8	.5	.2887	.75	.3660	.25
9	.5	.3536	.75	.3953	.25
10	.4	.2619	.6	.4	.25
11	.5	.3536	.75	.3953	.125
12	.5	.3536	.75	.3953	.5

TABLE 7

Prior Parameters Used in Calculation of Tables 8-12

5. MANY FAILURE MODES

5.1 NOTATIONAL EXTENSION

In order to treat the more realistic case of systems with multiple failure modes, we introduce a simple extended model and notation, and then show that this case is solved formally by a simple extension of previously obtained solutions. The development will be only for the continuous model, although a similar one for the discrete case can be directly obtained by means of a parallel analysis.

We now assume that a system can exhibit a total of M independent failure modes (characterized, by definition, by their distinguishability). We also assume that a repair of a mode is possible only at a repair attempt made after an observed failure of <u>that</u> mode.

We then define, for mode i $(i = 1, 2, \ldots M)$,

TRIAL NO.	CUM. SUCCESS	E(1-u)	σ(1-u)	E(1-v)	σ(1-v)	$P(R_N)$	E(p)	σ(p)
PRIOR		.5000	.2887	.7500	.3660	.2500		
1	0	.3333	.2357	.7500	.3660	.2500	.4375	.3282
2	1	.4286	.2433	.8265	.3101	.4286	.5991	.3374
3	1	.4055	.2114	.6591	.3901	.2364	.4654	.2858
4	2	.4749	.2080	.7244	.3467	.2204	.5299	.2663
5	2	.4208	.1932	.6792	.3574	.2676	.4899	.2732
6	3	.4680	.1951	.7085	.3351	.2927	.5384	.2679
7	3	.4334	.1865	.6573	.3467	.3372	.5089	.2735
8	4	.4661	.1894	.6852	.3269	.3523	.5433	.2680
9	5	.4901	.1957	.7036	.3122	.3811	.5715	.2675
10	6	.5066	.2037	.7186	.2997	.4149	.5945	.2692
11	6	.4772	.1940	.6854	.2991	.4539	.5717	.2682
12	7	.4895	.2009	.7010	.2867	.4840	.5919	.2679
13	8	.4977	.2081	.7151	.2755	.5157	.6098	.2682
14	9	.5025	.2149	.7283	.2651	.5477	.6262	.2683
15	9	.4849	.2068	.6948	.2642	.5682	.6041	.2625
16	10	.4894	.2125	.7078	.2545	.5935	.6190	.2613
17	11	.4920	.2177	.7199	.2454	.6186	.6330	.2600
18	12	.4932	.2224	.7315	.2369	.6430	.6464	.2584
19	12	.4771	.2149	.7037	.2315	.6734	.6297	.2500
20	13	.4781	.2185	.7157	.2234	.6941	.6430	.2474
21	14	.4783	.2215	.7270	.2156	.7141	.6559	.2447
22	15	.4779	.2241	.7378	.2083	.7332	.6685	.2417
23	15	.4676	.2198	.7092	.2041	.7517	.6492	.2328

TABLE 8

44

TRIAL NO.	CUM. SUCCESS	$E(1-u)$	$\sigma(1-u)$	$E(1-v)$	$\sigma(1-v)$	$P(R_N)$	$E(p)$	$\sigma(p)$
PRIOR		.5000	.3536	.7500	.3953	.2500		
1	0	.2500	.2500	.7500	.3953	.2500	.3750	.3644
2	1	.3750	.2795	.6542	.3131	.5000	.6146	.3814
3	1	.3804	.2316	.6487	.4251	.2174	.4388	.3058
4	2	.4680	.2286	.7296	.3779	.1739	.5135	.2788
5	2	.4016	.2074	.6925	.3861	.2075	.4620	.2810
6	3	.4612	.2120	.7161	.3674	.2291	.5196	.2776
7	3	.4165	.1997	.6750	.3766	.2664	.4853	.2830
8	4	.4570	.2054	.6974	.3599	.2800	.5243	.2798
9	5	.4865	.2141	.7120	.3477	.3042	.5551	.2818
10	6	.5073	.2241	.7241	.3374	.3319	.5792	.2860
11	6	.4702	.2130	.6961	.3358	.3692	.5536	.2866
12	7	.4859	.2220	.7086	.3252	.3949	.5739	.2889
13	8	.4970	.2311	.7201	.3157	.4219	.5911	.2916
14	9	.5044	.2397	.7308	.3068	.4494	.6061	.2943
15	9	.4784	.2310	.7025	.3041	.4758	.5850	.2907
16	10	.4849	.2385	.7132	.2956	.4984	.5987	.2917
17	11	.4892	.2454	.7233	.2876	.5210	.6112	.2926
18	12	.4918	.2516	.7230	.2801	.5434	.6228	.2932
19	12	.4682	.2437	.7084	.2726	.5811	.6078	.2865
20	13	.4704	.2487	.7185	.2651	.6009	.6195	.2858
21	14	.4716	.2531	.7282	.2579	.6203	.6308	.2848
22	15	.4720	.2569	.7375	.2511	.6393	.6418	.2835
23	15	.4540	.2520	.7121	.2442	.6672	.6262	.2752

TABLE 9

TRIAL NO.	CUM. SUCCESS	$E(1-u)$	$\sigma(1-u)$	$E(1-v)$	$\sigma(1-v)$	$P(R_N)$	$E(p)$	$\sigma(p)$
PRIOR		.4000	.2619	.6000	.4000	.2500		
1	0	.2857	.2130	.6000	.4000	.2500	.3643	.3042
2	1	.3791	.2262	.7098	.3611	.4118	.5153	.3321
3	1	.3650	.2015	.5222	.3897	.3169	.4148	.2850
4	2	.4212	.2052	.6053	.3591	.3114	.4786	.2765
5	2	.3814	.1891	.5547	.3559	.3010	.4440	.2748
6	3	.4185	.1962	.5983	.3373	.4013	.4907	.2765
7	3	.3946	.1863	.5420	.3347	.4428	.4599	.2726
8	4	.4187	.1932	.5806	.3186	.4695	.4947	.2720
9	5	.4344	.2024	.6104	.3053	.5094	.5241	.2755
10	6	.4432	.2120	.6369	.2933	.5525	.5502	.2773
11	6	.4235	.2000	.6008	.2848	.5827	.5269	.2676
12	7	.4298	.2075	.6262	.2736	.6183	.5512	.2680
13	8	.4327	.2141	.6496	.2630	.6535	.5745	.2679
14	9	.4332	.2197	.6715	.2530	.6871	.5969	.2670
15	9	.4232	.2116	.6334	.2461	.6978	.5699	.2552
16	10	.4241	.2162	.6534	.2373	.7236	.5901	.2534
17	11	.4238	.2200	.6721	.2289	.7479	.6095	.2511
18	12	.4227	.2230	.6896	.2209	.7705	.6283	.2482
19	12	.4137	.2152	.6606	.2124	.7885	.6084	.2356
20	13	.4132	.2173	.6775	.2051	.8067	.6264	.2323
21	14	.4122	.2188	.6934	.1982	.8236	.6438	.2287
22	15	.4110	.2198	.7083	.1915	.8392	.6605	.2246
23	15	.4051	.2163	.6780	.1848	.8484	.6366	.2137

TABLE 10

46

TRIAL NO.	CUM. SUCCESS	E(1-u)	σ(1-u)	E(1-v)	σ(1-v)	$P(R_N)$	E(p)	σ(p)
PRIOR		.5000	.3536	.7500	.3953	.1250		
1	0	.2500	.2500	.7500	.3953	.1250	.3125	.3187
2	1	.4250	.2750	.8125	.3520	.3000	.5412	.3488
3	1	.3784	.2236	.7015	.4137	.1009	.4110	.2678
4	2	.4868	.2153	.7409	.3879	.0747	.5058	.2420
5	2	.4112	.1949	.7265	.3928	.0846	.4375	.2357
6	3	.4848	.1922	.7362	.3844	.0948	.5086	.2299
7	3	.4308	.1788	.7203	.3905	.1032	.4607	.2284
8	4	.4848	.1777	.7295	.3831	.1083	.5114	.2232
9	5	.5266	.1787	.7351	.3778	.1199	.5516	.2232
10	6	.5589	.1819	.7398	.3732	.1342	.5832	.2281
11	6	.5165	.1712	.7299	.3746	.1412	.5466	.2247
12	7	.5435	.1751	.7344	.3698	.1556	.5732	.2280
13	8	.5649	.1803	.7388	.3652	.1720	.5948	.2327
14	9	.5819	.1865	.7431	.3607	.1897	.6125	.2385
15	9	.5526	.1750	.7323	.3634	.1841	.5857	.2327
16	10	.5684	.1806	.7362	.3592	.1991	.6018	.2373
17	11	.5813	.1868	.7401	.3551	.2150	.6154	.2424
18	12	.5916	.1933	.7440	.3509	.2318	.6269	.2477
19	12	.5658	.1838	.7346	.3507	.2387	.6060	.2454
20	13	.5750	.1899	.7386	.3462	.2558	.6169	.2502
21	14	.5824	.1961	.7426	.3417	.2739	.6263	.2550
22	15	.5880	.2023	.7468	.3372	.2927	.6345	.2597
23	15	.5707	.1928	.7357	.3393	.2860	.6179	.2550

TABLE 11

47

TRIAL NO.	CUM. SUCCESS	E(1-u)	σ(1-u)	E(1-v)	σ(1-v)	$P(R_N)$	E(p)	σ(p)
PRIOR		.5000	.3536	.7500	.3953	.5000		
1	0	.2500	.2500	.7500	.3953	.5000	.5000	.4146
2	1	.3125	.2724	.9062	.2466	.7500	.7578	.3609
3	1	.2750	.2500	.5269	.4186	.5000	.4560	.3541
4	2	.4065	.2568	.7005	.3390	.4634	.5427	.3318
5	2	.3613	.2417	.5917	.3413	.5833	.4957	.3243
6	3	.3622	.2522	.6565	.3041	.6165	.5513	.3149
7	3	.3551	.2491	.5540	.2923	.7316	.5006	.2949
8	4	.3614	.2558	.6116	.2652	.7502	.5491	.2843
9	5	.3654	.2627	.6528	.2444	.7725	.5874	.2763
10	6	.3670	.2685	.6854	.2278	.7940	.6198	.2695
11	6	.3445	.2621	.6200	.2093	.8650	.5828	.2367
12	7	.3447	.2649	.6522	.1963	.8764	.6142	.2295
13	8	.3445	.2672	.6794	.1851	.8870	.6415	.2230
14	9	.3440	.2688	.7027	.1754	.8966	.6656	.2168
15	9	.3304	.2649	.6514	.1605	.9352	.6306	.1868
16	10	.3299	.2657	.6742	.1528	.9400	.6535	.1813
17	11	.3295	.2662	.6943	.1459	.9444	.6740	.1762
18	12	.3291	.2666	.7121	.1397	.9483	.6923	.1714
19	12	.3215	.2639	.6718	.1296	.9691	.6610	.1487
20	13	.3212	.2639	.6893	.1246	.9711	.6786	.1445
21	14	.3211	.2638	.7050	.1200	.9729	.6947	.1406
22	15	.3211	.2637	.7194	.1157	.9746	.7092	.1369
23	15	.3165	.2628	.6854	.1094	.9852	.6799	.1217

TABLE 12

λ_i = failure rate when i^{th} mode is unrepaired

μ_i = failure rate when i^{th} mode is repaired

a_i = probability of repairing the i^{th} mode given an attempt is made

The entire system will have an overall failure rate r, which, by virtue of the exponential failure behavior of each component, is

$$r = \sum_{i=1}^{M} r_i$$

where

$$r_i = \begin{cases} \lambda_i & i^{th} \text{ mode is unrepaired} \\ \mu_i & i^{th} \text{ mode is repaired} \end{cases}$$

This last expression serves to recall that, according to our previous analysis, the failure rates are in themselves random variables.

If, then, the failure rate for each mode is a random variable $\underline{r}_i$, with known p.d.f. $f_{\underline{r}_i}(r_i)$ [and thus known moments], we have in particular for the overall system

$$f_{\underline{r}}(r) = f_{\underline{r}_1}(r_1) * f_{\underline{r}_2}(r_2) * \ldots f_{\underline{r}_M}(r_M) \tag{75}$$

where the * indicates the convolution operation.

Because of the independence of the failure modes, and since the repair of any one mode is independent of the state of the others, we see that each of the $f_{\underline{r}_i}(r_i)$ of equation (75) is available from expressions such as (31) [for projection] or (40) [for inference]. In these expressions we must only

replace the parameters (r, λ, μ, a) by $(r_i, \lambda_i, \mu_i, a_i)$, and note that $\vec{t}$ now represents the times of occurrences of i^{th} mode failures.

To make matters even simpler for practical purposes, we note that since $\underline{r} = \Sigma \underline{r}_i$, and the $\underline{r}_i$ are independent, we can immediately write for the expectation and variances:

$$E(\underline{r}) = \sum_{i=1}^{M} E(\underline{r}_i)$$

$$V(\underline{r}) = \sigma^2(\underline{r}) = \sum_{i=1}^{M} \sigma^2(\underline{r}_i)$$

6. CONCLUSION

6.1 OTHER MODELS OF RELIABILITY GROWTH

Discussion of the literature on reliability growth models has been intentionally postponed to this final section in order to facilitate comparison with this paper.

The subject of reliability improvement by means of conscious efforts on the part of designers, test engineers, customers, etc. has been of interest from the beginnings of reliability analysis. The modelling of such growth processes has followed, for the most part, a common procedure: formulae are presented that are intended to represent the growth of reliability (or the decrease in failure rate, etc.) as a function of time. These formulae contain unknown parameters, and it becomes a statistical problem to find appropriate estimates (and confidence statements) for these parameters as a

function of observed failure data. Such methods are found, for example, in references [10], [3], [15] and [8]. Sherman [14], for example, finds Maximum Likelihood Estimates for the repair probability a and the unrepaired failure probability u when it is assumed that the repaired failure probability v is zero.

Another approach is to assume that little is known about the underlying failure behavior of the system, and what amounts to "almost" non-parametric analysis is made upon eventual failure rates (or probabilities). This is summarized in [1].

Bayesian techniques have been used only recently. A non-parametric Bayesian analysis of a failure probability, constrained to be only non-increasing in time, may be modelled by the technique shown in Samuels [13]. Larson [9] has extended an earlier analysis [8] to produce Bayesian estimates of parameters of a growth model, using prior distributions suggested by Earnest [5]. Finally, Cozzolino [4] has presented a Bayesian approach to a general class of growth models with regard to making minimum-cost decisions about length of tests and burn-in procedures.

All of the above analyses, however, start with a basic assumption: that the reliability <u>will</u> grow (or, at least, will not decrease) in time. If the techniques derived previously were to be used for a system that was actually deteriorating (naturally, or because of well-intentioned intervention), the results would be meaningless. In practice, unfortunately, there is often

a need to have an inferential technique that would spot such deterioration, as well as one equally good at determining appropriate growth characteristics.

6.2 CONCLUSION

This paper has attempted to model a process that simply considers a system (with regard to each failure mode) to be in either a repaired or unrepaired state. The failure rates in each state are known to any desired degree of confidence, and accumulation of failure data serves, in a natural way, to update the knowledge of these state parameters. The observation of failure data also determines the probability that the system is repaired (with respect to each mode).

The weakest points of the model seem to be the assumptions that

- . The repair probability a is known
- . Repair attempts occur only after the observation of a failure

The first point can be overcome (at the expense of additional complexity) by considering a to be a random variable $\underline{a}$ with appropriate prior p.d.f. $f_{\underline{a}}(a|H)$. All analysis would then include a posterior inferential p.d.f. for $\underline{a}$, given a data vector.

The second point is unfortunately too much at the heart of the model. For many realistic systems, the assumption seems to be valid, however, as the tendency is not to "ruin a good thing".

It should be pointed out that the model considered here is a specific example of a process which Howard [6] calls "Dynamic Inference". This general concept is quite useful in modelling a stochastic process in which the underlying parameters are allowed to change according to yet another stochastic process. The interested reader is referred to reference [6], where (as becomes apparent upon studying the Tables 2-6 and 8-12) the statement is made, "The numerical results indicate a complexity of behavior that challenges intuition".

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13. ABSTRACT

A model is presented for the reliability growth of a system during a test program. Parameters of the model are assumed to be random variables with appropriate prior density functions. Expressions are then derived that enable estimates (in the form of expectations) and precision statements (in the form of variances) to be made of

- projected system reliability at time τ after the start of the test program
- system reliability after the observation of failure data

Numerical examples are presented, and extension to multi-mode failures is mentioned.

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